1.12 Spatial Invariants

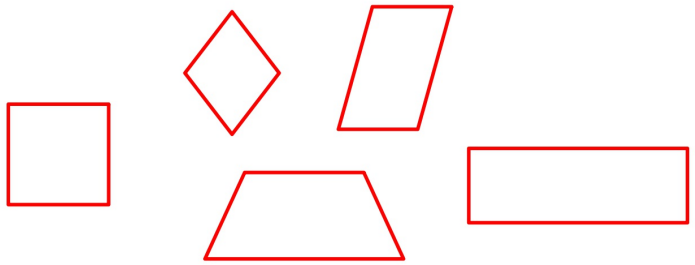
Objectives:

- Describe various types of invariants in geometry
- Search for geometric invariants, such as points of concurrency and collinearity of points.

Invariants do not have to be numbers or relationships between numbers.

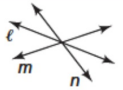
Invariants can be shapes or relationships between shapes as well

What is invariant of the following group of shapes?



Definition

Three or more lines that meet or intersect at one point are **concurrent lines.**



Lines ℓ, m, and n are concurrent lines.

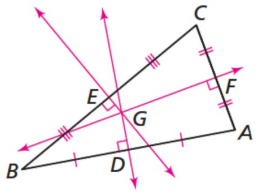
For You to Do

4. Use geometry software to draw a triangle. Construct the perpendicular bisector of each side. Can you adjust the triangle so that the three perpendicular bisectors are concurrent?
5. Hide the perpendicular bisectors. Construct the angle bisectors of your triangle. Can you adjust the triangle so that the three angle bisectors are concurrent?
6. Under which circumstances, if any, are the three perpendicular bisectors and the three angle bisectors of a triangle concurrent?

Theorem 1.2 Concurrence of Perpendicular Bisectors

In any triangle, the perpendicular bisectors of the sides are concurrent.

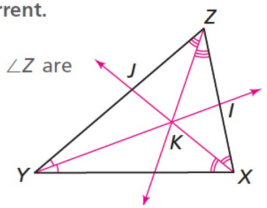
In $\triangle ABC$, the perpendicular bisectors of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ are concurrent at G.



Theorem 1.3 Concurrence of Angle Bisectors

In any triangle, the angle bisectors are concurrent.

In $\triangle XYZ$, the angle bisectors of $\angle X$, $\angle Y$, and $\angle Z$ are concurrent at K.



Definition

Three or more points that are on the same line are **collinear points.**

For You to Do

Step 1 Trace a circle onto a sheet of paper.

Step 2 Poke a small hole through the paper at the center of the circle.

Step 3 Carefully cut out your **disc** (the circle and its interior).

Step 4 Work with three or four classmates who made discs of different sizes. Draw two points on a large sheet of paper. Place them close enough together that the smallest disc can touch both.

Step 5 Place one of your discs on the large sheet. Move it so that the edge of the disc touches both points.

Step 6 Mark the center of the circle on the large sheet.

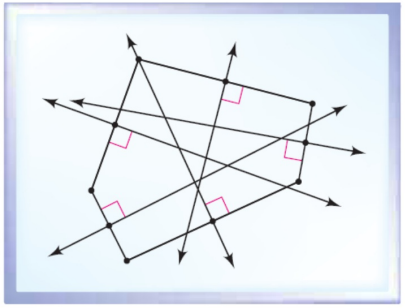
Step 7 Remove the disc. Repeat Steps 5 and 6 with the other discs.

Step 8 After using all your discs, look at the circles' center marks. What is invariant about their positions?

Step 9 Draw two new points. Without using your discs, draw the figure that would be formed by a large number of circle center marks.

Check Your Understanding

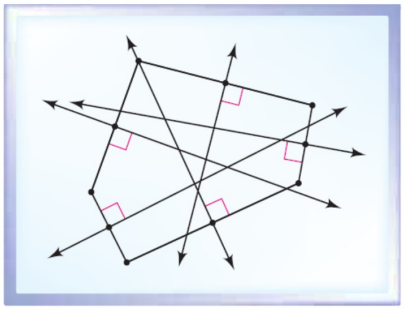
1. Use geometry software. Place five points on your screen. Connect them with segments so that you have an arbitrary convex pentagon. Construct the perpendicular bisector of each side of your pentagon. See the diagram and answer the questions on the next page.

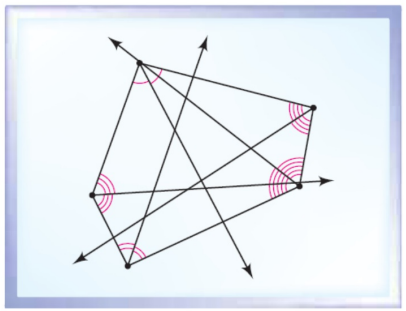


a. In the diagram at the left below, is there a point at which three or more perpendicular bisectors are concurrent?

b. If not, is it possible to adjust the vertices of the pentagon so that at least three bisectors are concurrent?

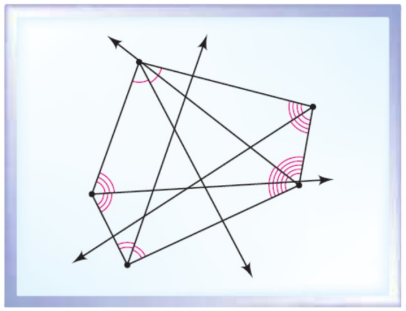
c. Is it possible to adjust the vertices of the pentagon so that all five bisectors are concurrent?





2. Try the same kind of experiment with the angle bisectors of a pentagon. See the diagram at the right above. Start with an arbitrary pentagon. Construct the angle bisector of each angle. Is it possible to adjust the vertices of the pentagon so that all five angle bisectors are concurrent?

It is a good habit to start an investigation with a special case. It simplifies what you have to look at. It can also suggest what to look for in other cases. Among polygons, the triangle is special because it is the simplest. Exercises 3–6 suggest other special cases.



3. Regular polygons are a very special case. Is concurrence of perpendicular bisectors an invariant for regular polygons? Experiment. Be sure to experiment with regular quadrilaterals (squares), regular pentagons, and regular hexagons. What do your experiments suggest? Explain.

4. Is concurrence of angle bisectors an invariant for regular polygons? Experiment, describe a conjecture, and explain the result.

5. Use geometry software to construct a circle. Place five points on the circle. Connect the points to form an irregular pentagon. Check the angle bisectors and perpendicular bisectors of the pentagon for concurrence. Do you observe any invariants? Explain.

6. Draw a circle. Construct an irregular polygon outside the circle so that all the sides of the polygon are tangent to the circle. Perform the two concurrence experiments. Do you observe any invariants? Explain.

On Your Own

Classwork: p.63 (7, 8)

7. Construct several different triangles. Then construct their medians. Describe any concurrence or collinearity you find.

8. **Standardized Test Prep** Triangle ABC is an isosceles triangle. $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are altitudes. $\overline{AD}$, $\overline{BG}$, and $\overline{CH}$ are angle bisectors. Points D, I, and J are the midpoints of $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively.

   Which of the following statements may NOT be true?

   A. The concurrences of the altitudes, angle bisectors, and medians are collinear.

   B. $\overline{CI} = \overline{CD}$

   C. $\overline{AD}$ is a median.

   D. $\angle BCH = \angle HCA$